General Certificate of Education
Advanced Level Examination
January 2011

# Mathematics MFP3

**Unit Further Pure 3**

**Monday 24 January 2011 9.00 am to 10.30 am**

**For this paper you must have:**
- the blue AQA booklet of formulae and statistical tables.

You may use a graphics calculator.

**Time allowed**
- 1 hour 30 minutes

**Instructions**
- Use black ink or black ball-point pen. Pencil should only be used for drawing.
- Fill in the boxes at the top of this page.
- Answer **all** questions.
- Write the question part reference (eg (a), (b)(i) etc) in the left-hand margin.
- You must answer the questions in the spaces provided. Do not write outside the box around each page.
- Show all necessary working; otherwise marks for method may be lost.
- Do all rough work in this book. Cross through any work that you do not want to be marked.

**Information**
- The marks for questions are shown in brackets.
- The maximum mark for this paper is 75.

**Advice**
- Unless stated otherwise, you may quote formulae, without proof, from the booklet.

**1** The function $y(x)$ satisfies the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \mathrm{f}(x, y)$$

where $$\mathrm{f}(x, y) = x + \sqrt{y}$$

and $$y(3) = 4$$

Use the improved Euler formula

$$y_{r+1} = y_r + \tfrac{1}{2}(k_1 + k_2)$$

where $k_1 = h\mathrm{f}(x_r, y_r)$ and $k_2 = h\mathrm{f}(x_r + h, y_r + k_1)$ and $h = 0.1$, to obtain an approximation to $y(3.1)$, giving your answer to three decimal places. *(5 marks)*

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**2 (a)** Find the values of the constants $p$ and $q$ for which $p\sin x + q\cos x$ is a particular integral of the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} + 5y = 13\cos x$$ *(3 marks)*

**(b)** Hence find the general solution of this differential equation. *(3 marks)*

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**3** A curve $C$ has polar equation $r(1 + \cos\theta) = 2$.

**(a)** Find the cartesian equation of $C$, giving your answer in the form $y^2 = \mathrm{f}(x)$. *(5 marks)*

**(b)** The straight line with polar equation $4r = 3\sec\theta$ intersects the curve $C$ at the points $P$ and $Q$. Find the length of $PQ$. *(4 marks)*

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**4** By using an integrating factor, find the solution of the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} - \frac{2}{x}y = 2x^3\mathrm{e}^{2x}$$

given that $y = \mathrm{e}^4$ when $x = 2$. Give your answer in the form $y = \mathrm{f}(x)$. *(9 marks)*

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**5 (a)** Write $\dfrac{4}{4x+1} - \dfrac{3}{3x+2}$ in the form $\dfrac{C}{(4x+1)(3x+2)}$, where $C$ is a constant. *(1 mark)*

**(b)** Evaluate the improper integral

$$\int_1^{\infty} \frac{10}{(4x+1)(3x+2)}\,\mathrm{d}x$$

showing the limiting process used and giving your answer in the form $\ln k$, where $k$ is a constant. *(6 marks)*

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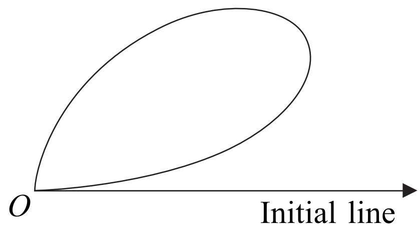
**6** The diagram shows a sketch of a curve $C$.




The polar equation of the curve is

$$r = 2\sin 2\theta\sqrt{\cos\theta}, \quad 0 \leqslant \theta \leqslant \frac{\pi}{2}$$

Show that the area of the region bounded by $C$ is $\frac{16}{15}$. *(7 marks)*

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**7 (a)** Write down the expansions in ascending powers of $x$ up to and including the term in $x^3$ of:

**(i)** $\cos x + \sin x$; *(1 mark)*

**(ii)** $\ln(1+3x)$. *(1 mark)*

**(b)** It is given that $y = \mathrm{e}^{\tan x}$.

**(i)** Find $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ and show that $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2} = (1+\tan x)^2\dfrac{\mathrm{d}y}{\mathrm{d}x}$. *(5 marks)*

**(ii)** Find the value of $\dfrac{\mathrm{d}^3y}{\mathrm{d}x^3}$ when $x = 0$. *(2 marks)*

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**(iii)** Hence, by using Maclaurin's theorem, show that the first four terms in the expansion, in ascending powers of $x$, of $e^{\tan x}$ are

$$1 + x + \tfrac{1}{2}x^2 + \tfrac{1}{2}x^3$$ *(2 marks)*

**(c)** Find

$$\lim_{x \to 0} \left[ \frac{e^{\tan x} - (\cos x + \sin x)}{x \ln(1 + 3x)} \right]$$ *(3 marks)*

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**8 (a)** Given that $x = e^t$ and that $y$ is a function of $x$, show that

$$x \frac{dy}{dx} = \frac{dy}{dt}$$ *(2 marks)*

**(b)** Hence show that the substitution $x = e^t$ transforms the differential equation

$$x^2 \frac{d^2y}{dx^2} - 3x\frac{dy}{dx} + 4y = 2\ln x$$

into

$$\frac{d^2y}{dt^2} - 4\frac{dy}{dt} + 4y = 2t$$ *(5 marks)*

**(c)** Find the general solution of the differential equation

$$\frac{d^2y}{dt^2} - 4\frac{dy}{dt} + 4y = 2t$$ *(6 marks)*

**(d)** Hence solve the differential equation $x^2 \dfrac{d^2y}{dx^2} - 3x\dfrac{dy}{dx} + 4y = 2\ln x$, given that $y = \dfrac{3}{2}$ and $\dfrac{dy}{dx} = \dfrac{1}{2}$ when $x = 1$. *(5 marks)*