General Certificate of Education
Advanced Level Examination
January 2012

Mathematics MFP2

Unit Further Pure 2

Friday 20 January 2012 1.30 pm to 3.00 pm

For this paper you must have:
- the blue AQA booklet of formulae and statistical tables.

You may use a graphics calculator.

Time allowed
- 1 hour 30 minutes

Instructions
- Use black ink or black ball-point pen. Pencil should only be used for drawing.
- Fill in the boxes at the top of this page.
- Answer **all** questions.
- Write the question part reference (eg (a), (b)(i) etc) in the left-hand margin.
- You must answer the questions in the spaces provided. Do not write outside the box around each page.
- Show all necessary working; otherwise marks for method may be lost.
- Do all rough work in this book. Cross through any work that you do not want to be marked.

Information
- The marks for questions are shown in brackets.
- The maximum mark for this paper is 75.

Advice
- Unless stated otherwise, you may quote formulae, without proof, from the booklet.
- You do not necessarily need to use all the space provided.

1 (a) Show, by means of a sketch, that the curves with equations

$$y = \sinh x$$

and $$y = \operatorname{sech} x$$

have exactly one point of intersection. *(4 marks)*

(b) Find the x-coordinate of this point of intersection, giving your answer in the form $a \ln b$. *(4 marks)*

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2 (a) Draw on an Argand diagram the locus L of points satisfying the equation $\arg z = \dfrac{\pi}{6}$. *(1 mark)*

(b) (i) A circle C, of radius 6, has its centre lying on L and touches the line $\operatorname{Re}(z) = 0$. Draw C on your Argand diagram from part **(a)**. *(2 marks)*

(ii) Find the equation of C, giving your answer in the form $|z - z_0| = k$. *(3 marks)*

(iii) The complex number z_1 lies on C and is such that $\arg z_1$ has its least possible value. Find $\arg z_1$, giving your answer in the form $p\pi$, where $-1 < p \leqslant 1$. *(2 marks)*

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3 A curve has cartesian equation

$$y = \tfrac{1}{2}\ln(\tanh x)$$

(a) Show that

$$\frac{dy}{dx} = \frac{1}{\sinh 2x}$$ *(4 marks)*

(b) The points A and B on the curve have x-coordinates $\ln 2$ and $\ln 4$ respectively. Find the arc length AB, giving your answer in the form $p \ln q$, where p and q are rational numbers. *(8 marks)*

4 The sequence u_1, u_2, u_3, … is defined by

$$u_1 = \frac{3}{4} \qquad u_{n+1} = \frac{3}{4 - u_n}$$

Prove by induction that, for all $n \geqslant 1$,

$$u_n = \frac{3^{n+1} - 3}{3^{n+1} - 1}$$ *(6 marks)*

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5 Find the smallest positive integer values of p and q for which

$$\frac{\left(\cos\frac{\pi}{8} + \mathrm{i}\sin\frac{\pi}{8}\right)^p}{\left(\cos\frac{\pi}{12} - \mathrm{i}\sin\frac{\pi}{12}\right)^q} = \mathrm{i}$$ *(7 marks)*

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6 (a) Express $7 + 4x - 2x^2$ in the form $a - b(x - c)^2$, where a, b and c are integers. *(2 marks)*

(b) By means of a suitable substitution, or otherwise, find the exact value of

$$\int_1^{\frac{5}{2}} \frac{\mathrm{d}x}{\sqrt{7 + 4x - 2x^2}}$$ *(6 marks)*

Turn over ▶

7 The numbers α, β and γ satisfy the equations

$$\alpha^2 + \beta^2 + \gamma^2 = -10 - 12\mathrm{i}$$
$$\alpha\beta + \beta\gamma + \gamma\alpha = 5 + 6\mathrm{i}$$

(a) Show that $\alpha + \beta + \gamma = 0$. *(2 marks)*

(b) The numbers α, β and γ are also the roots of the equation

$$z^3 + pz^2 + qz + r = 0$$

Write down the value of p and the value of q. *(2 marks)*

(c) It is also given that $\alpha = 3\mathrm{i}$.

(i) Find the value of r. *(3 marks)*

(ii) Show that β and γ are the roots of the equation

$$z^2 + 3\mathrm{i}z - 4 + 6\mathrm{i} = 0$$ *(2 marks)*

(iii) Given that β is real, find the values of β and γ. *(3 marks)*

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8 (a) Write down the five roots of the equation $z^5 = 1$, giving your answers in the form $\mathrm{e}^{\mathrm{i}\theta}$, where $-\pi < \theta \leqslant \pi$. *(1 mark)*

(b) Hence find the four linear factors of

$$z^4 + z^3 + z^2 + z + 1$$ *(3 marks)*

(c) Deduce that

$$z^2 + z + 1 + z^{-1} + z^{-2} = \left(z - 2\cos\frac{2\pi}{5} + z^{-1}\right)\left(z - 2\cos\frac{4\pi}{5} + z^{-1}\right)$$ *(4 marks)*

(d) Use the substitution $z + z^{-1} = w$ to show that $\cos\frac{2\pi}{5} = \frac{\sqrt{5} - 1}{4}$. *(6 marks)*